Lecture 13

Note Title 5/28/2009

Ekman transport, pumping/suction, and the spin-down of geostrophic flows

Last time we began to look at the effects of friction on oceanic flows. Frictional effects are critically important for the ocean circulation because they are responsible for both its spin-down or decelleration as well as its acelleration or spin-up by the winds.

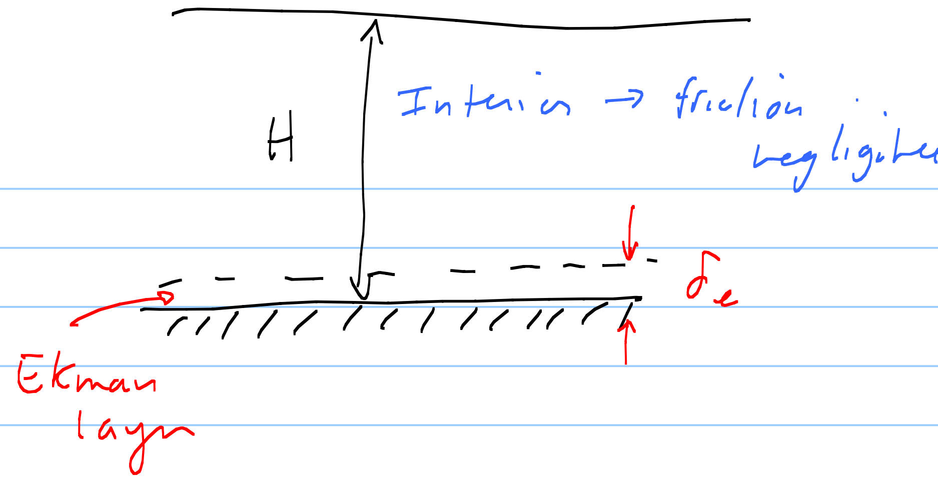
We saw that the relative strength of friction to the Coriolis force was characterized by the non-dimensional parameter

$$Ek = \frac{\nu_{ed}}{f H^2}$$

The Ekman number

Which for typical flow parameters is much less than 1.

This is not to say that friction is not unimportant, we found that while for most of the water column the direct effects of friction are negligible, in a boundary layer, known as the Ekman layer, friction is as important as the Coriolis force.



We saw that the Ekman layer had a thickness

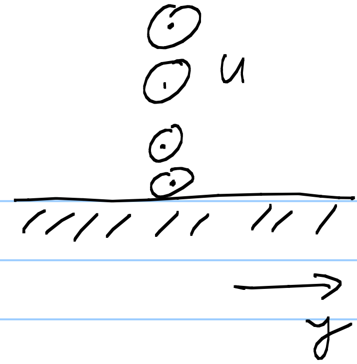
$$\delta_e = \sqrt{\frac{2\nu_{ed}}{f}}$$

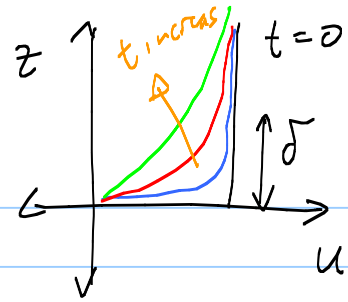
Notice that the Ekman number is just

$$Ek = \frac{1}{2}\left(\frac{\delta_e}{H}\right)^2$$

→ When the Ekman layer is much thinner than the total depth of the water column then friction can be neglected in most of the watercolumn.

What is the physical interpretation of the Ekman layer thickness? To answer this think about what happens to the spin-down of a flow in the non-rotating limit





at $z=0$ $u=0$ for $t>0$

As time increase a boundary layer near the bottom grows diffusively as

$$\delta = \sqrt{\nu_{ed}\, t}$$

Compare this expression to the Ekman layer thickness

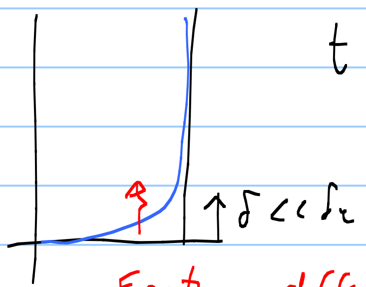
$$\delta_e = \sqrt{\frac{2\nu_{ed}}{f}}$$

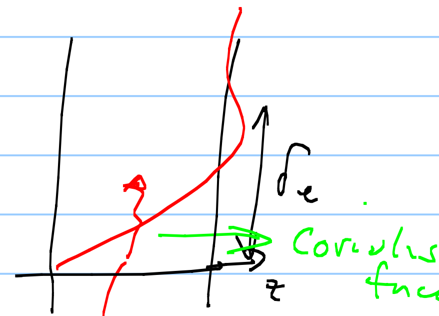
→ Interpret the Ekman layer thickness as the thickness that a diffusively growing boundary layer would obtain in a time

$$T = \frac{2}{f} = \frac{2\pi}{f}\frac{1}{\pi} = \frac{T_i}{\pi}$$

i.e. the thickness that a diffusively growing boundary layer grows to in about an inertial period.

Recall that an inertial period is the time it takes for the fluid to feel the effects of rotation, so that for a rotating fluid :





Within about an inertial period the Coriolis force balances the diffusion of momentum, a balance is attained an the growth of the boundary layer is arrested.

We saw that the structure of the flow in the Ekman layer takes the shape of a spiral :

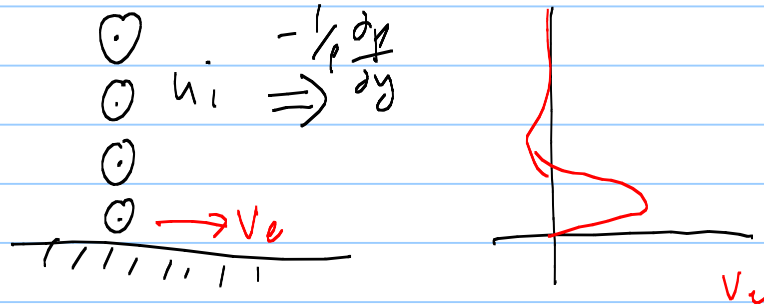
$$u_e = -V_i e^{-z/\delta_e} \sin(z/\delta_e) - u_i e^{-z/\delta_e} \cos(z/\delta_e)$$

$$v_e = u_i e^{-z/\delta_e} \sin(z/\delta_e) - V_i e^{-z/\delta_e} \cos(z/\delta_e)$$

Where $u_i = u_g = -\frac{1}{\rho_0 f} \frac{\partial p}{\partial y}$

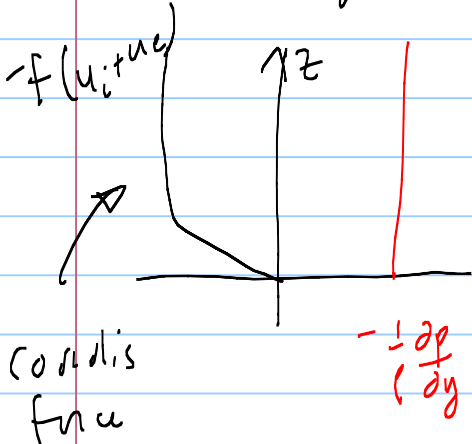
$$V_i = V_g = \frac{1}{\rho_0 f} \frac{\partial p}{\partial x}$$

For the case where $\frac{\partial p}{\partial x} = 0$ $\quad -\frac{1}{\rho_0}\frac{\partial p}{\partial y} \neq 0 > 0$

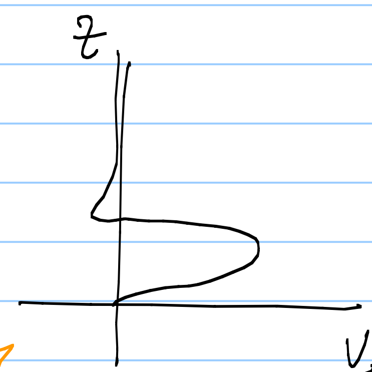


The Ekman flow (unlike the geostrophic) flow has a component directed down the pressure gradient. Why is this.

In the Ekman layer, owing to the no-slip boundary condition, the southward Coriolis force goes to zero near the bottom and thus cannot balance the pressure gradient force:



In b.l. excess PGF generates a northward flow whose friction force balances the PGF



$\Rightarrow$ Similar to steady pipe flow,

PGF + FRICTION = 0

Like a pipe, flow in the Ekman transport water from high to low pressure.

How much water does the Ekman flow transport?

The net transport is:

$$M_e^x \equiv \int_0^{z_t} u_e \, dz \qquad M_e^y = \int_0^{z_t} v_e \, dz$$

where $z_t \gg \delta_e$. $\qquad \vec{M}_e = (M_e^x, M_e^y)$

We could just integrate the Ekman spiral to calculate the Ekman transport but we can also get it from the force balance in the Ekman layer:

$$f\hat{k} \times \vec{u}_e = \frac{\partial}{\partial z}\left(\frac{\vec{\tau}}{\rho_0}\right)$$

Vertically integrating this relation between $z=0$ to $z=z_t$:

$$f\hat{k} \times \int_0^{z_t} \vec{u}_e \, dz = f\hat{k} \times \vec{M}_e = \frac{\vec{\tau}}{\rho_0}\Bigg|_{z=0}^{z=z_t}$$

Since $z_t \gg \delta_e$ is in the interior where friction is small it is assumed that

$$\vec{\tau}\Big|_{z=z_t} = 0$$

$$f\hat{k}\times\vec{M}_e = -\frac{\vec{\tau}}{\rho_0}\Big|_{z=0}$$

or $$\vec{M}_e = -\frac{\vec{\tau}\times\hat{k}}{\rho_0 f}\Big|_{z=0}$$

For the Ekman spiral solution you will recall that we parameterized the stress as:

$$\vec{\tau} = \rho_0 \nu_{ed}\frac{\partial\vec{u}}{\partial z} = \rho_0 \nu_{ed}\frac{\partial\vec{u}_e}{\partial z}$$

From the solution:

$$u_e = -v_i\, e^{-z/\delta_e}\sin(z/\delta_e) - u_i\, e^{-z/\delta_e}\cos(z/\delta_e)$$

$$v_e = u_i\, e^{-z/\delta_e}\sin(z/\delta_e) - v_i\, e^{-z/\delta_e}\cos(z/\delta_e)$$

$$\left.\frac{\partial u_e}{\partial z}\right|_{z=0} = -\frac{v_i}{\delta_e} + \frac{u_i}{\delta_e}$$

$$\left.\frac{\partial v_e}{\partial z}\right|_{z=0} = \frac{u_i}{\delta_e} + \frac{v_i}{\delta_e}$$

$$\left.\frac{\vec{\tau}}{\rho_0}\right|_{z=0} = \frac{\nu}{\delta_e}\left[(u_i - v_i)\,\hat{i} + (u_i + v_i)\,\hat{j}\right]$$

$$\vec{M}_e = -\frac{\vec{\tau}\times\hat{k}}{\rho f} = -\frac{\nu}{f\delta_e}\left[(u_i - v_i)\,\hat{i}\times\hat{k} + (u_i + v_i)\,\hat{j}\times\hat{k}\right]$$

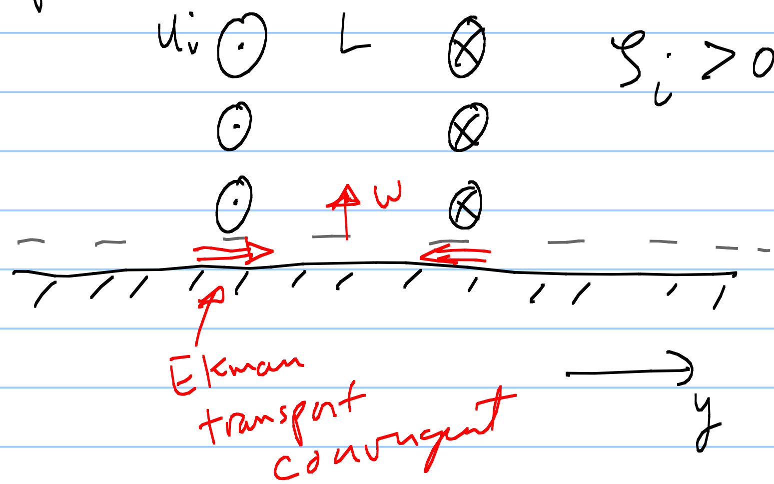
$$\boxed{\vec{M}_e = \frac{\delta_e}{2}\left[(u_i - v_i)\,\hat{j} - (u_i + v_i)\,\hat{i}\right]}$$

Therefore there is a component of the net transport $\perp$ to the geostrophic flow, directed down the pressure gradient since:

$$\vec{M}_e = \frac{\delta_e}{2}\left[\left(-\frac{1}{\rho_0 f}\frac{\partial p}{\partial y} - \frac{1}{\rho_0 f}\frac{\partial p}{\partial x}\right)\hat{j} + \left(\frac{1}{\rho_0 f}\frac{\partial p}{\partial y} - \frac{1}{\rho f}\frac{\partial p}{\partial x}\right)\hat{i}\right]$$

$$\boxed{\vec{M}_e = \frac{\delta_e}{2f}\left[-\frac{1}{\rho_0}\nabla_h p + \frac{1}{\rho_0}\nabla_h p \times \hat{k}\right]}$$

Now what if the geostrophic flow has spatial variations; for example



If the geostrophic flow has lateral shear, i.e. if it has vorticity, then the Ekman transport will be convergent or divergent and vertical motions will be generated.

Quantifying this vertical velocity.

For a spatially varying geostrophic flow we have the potential for generating vertical motions. This vertical velocity can be split up into interior and Ekman layer components:

$$w = w_i + w_e$$

Where as with the horizontal velocity
w_i has a characteristic vertical scale H & is found throughout the water column

while w_e has a characteristic scale δ_e and is confined to the Ekman layer

Thus the continuity equation becomes

$$\underbrace{\frac{\partial u_i}{\partial x} + \frac{\partial v_i}{\partial y} + \frac{\partial w_i}{\partial z}}_{\text{These terms have a vertical structure that extends through the water column}} + \underbrace{\frac{\partial u_e}{\partial x} + \frac{\partial v_e}{\partial y} + \frac{\partial w_e}{\partial z}}_{\text{These terms have a vertical structure confined to Ekman layer}} = 0$$

Because of the disparate vertical structures of these terms they must be considered seperately i.e.:

$$\text{(A)} \quad \frac{\partial u_e}{\partial x} + \frac{\partial v_e}{\partial y} + \frac{\partial w_e}{\partial z} = 0$$

$$\text{(B)} \quad \frac{\partial u_i}{\partial x} + \frac{\partial v_i}{\partial y} + \frac{\partial w_i}{\partial z} = 0$$

The boundary condition on the vertical velocity is:

$$w = w_i + w_e = 0 \quad \text{at } z = 0$$

i.e. the no-normal flow boundary condition

$$\rightarrow \quad w_i = -w_e \Big|_{z=0}$$

We can calculate $w_e \big|_{z=0}$ by integrating (A) in the vertical:

$$w_e \Big|_0^{z_t} = -\int_0^{z_t} \left(\frac{\partial u_e}{\partial x} + \frac{\partial v_e}{\partial y} \right) dz$$

Since $w_e \big|_{z=z_t} = 0$

$$w_e \Big|_{z=0} = + \nabla_h \cdot \vec{M}_e$$

$$\boxed{w_i \Big|_{z=0} = - \nabla_h \cdot \vec{M}_e}$$

Ekman pumping / suction from the bottom Ekman layer

When $w_i > 0 \rightarrow$ water going from the boundary layer to the interior this is called Ekman pumping

$w_i < 0 \rightarrow$ water going from the interior to the boundary layer $\rightarrow$ Ekman suction

Given our expression for $\vec{M}_e$ what is w_i in terms of the pressure and geostrophic flow?

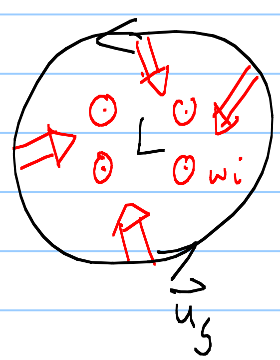
$$\vec{M}_e = \frac{\delta_e}{2f}\left[-\frac{1}{\rho_0}\nabla_h p + \frac{1}{\rho_0}\nabla_h p \times \hat{k}\right]$$

$$-\nabla\cdot\vec{M}_e = \frac{\delta_e}{2f}\left[+\frac{1}{\rho_0}\nabla_h^2 p\right]$$

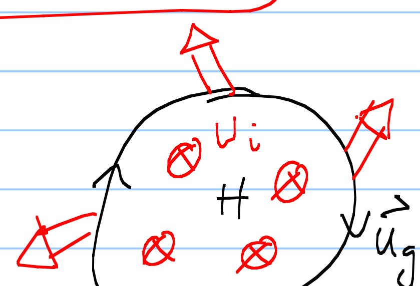
You will recall that

$$\frac{1}{f\rho_0}\nabla_h^2 p = \zeta_g = \frac{\partial v_g}{\partial x} - \frac{\partial u_g}{\partial y}$$

$$\Rightarrow \boxed{w_i\Big|_{z=0} = \frac{\delta_e}{2}\zeta_g}$$



CYCLONES
PRODUCE
EKMAN PUMPING



ANTICYCLONE
PRODUCE
EKMAN SUCTION

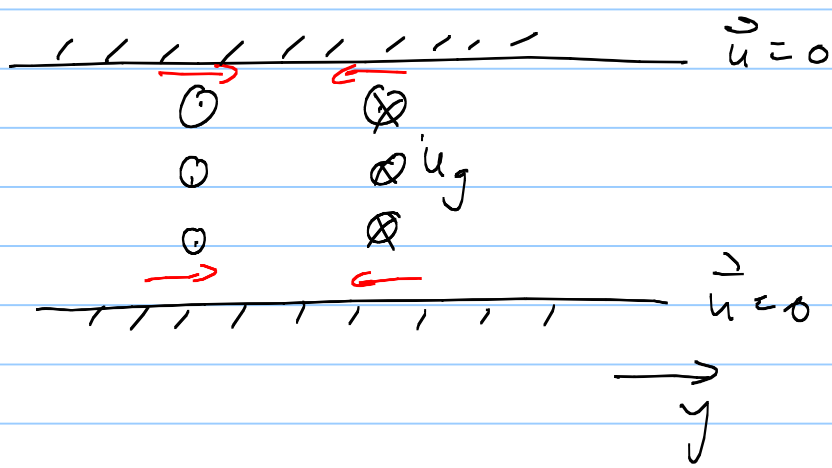
The Ekman flow transports water into the center of cyclones filling in the low pressure → expect it reduces the pressure gradient and hence spins down the flow.

In an anticyclone the Ekman flow transports water away from its center and hence would reduce the pressure gradient as well.

So you see that in both cases the Ekman flow will tend to spin-down or decelerate the flow.

Showing this quantitatively using a concrete example.

Consider a geostrophic flow between two solid, no-slip boundaries



You can see that an Ekman layer will develop at the top of the layer, and it can be shown that the

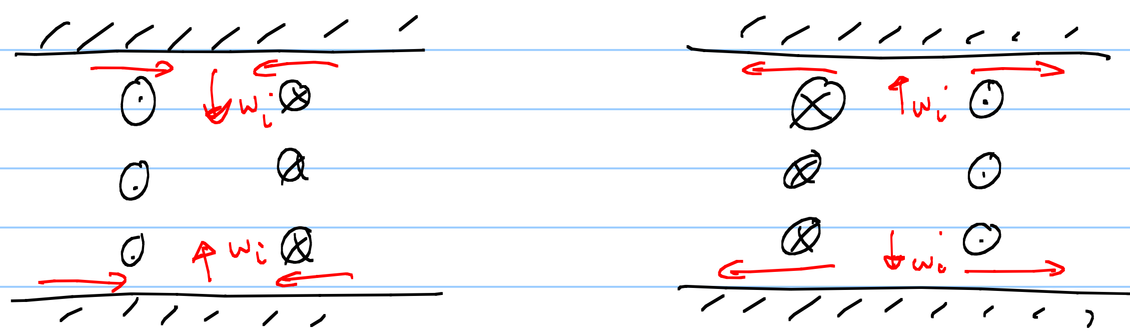
Ekman pumping/suction coming out of/into the ~~the~~ boundary layer is:

$$w_i\Big|_{z=H} = -\frac{\delta_e}{2}\zeta_g$$

The vorticity equation in the interior for low Rossby number flows is

$$\frac{\partial \zeta_g}{\partial t} = f\frac{\partial w_i}{\partial z}$$

i.e. the interior vorticity is changed by vortex squashing or stretching



CYCLONE
Ekman flow induces vortex squashing → reduces magnitude of vorticity

ANTICYCLONE
Ekman flow induces vortex stretching → reduces magnitude of vorticity

What is the vertical structure of w_i.
In the interior the geostrophic flow is independent of depth, therefore from the vorticity equation,

$$\frac{\partial \zeta_g}{\partial t} = f \frac{d w_i}{d z} \rightarrow \frac{\partial w_i}{\partial z} = \text{constant w.r.t } z$$

We can find this constant from the bcs on w_i:

$$w_i\big|_{z=0} = \frac{\delta_e}{2} \zeta_g$$

$$w_i\big|_{z=H} = -\frac{\delta_e}{2} \zeta_g$$

$$\Rightarrow \quad \frac{\partial w_i}{\partial z} = -\frac{\delta_e \zeta_g}{H}$$

Thus the vorticity equation becomes:

$$\frac{\partial \zeta_g}{\partial t} = -\frac{f \delta_e}{H} \zeta_g$$

The solution of which is simply:

$$\zeta_g = \zeta_g(t=0) \exp\left[-\left(\frac{\delta_e}{H}\right) f t\right]$$

You can see that the vorticity spins-down as we expected over an e-folding time:

$$T_{spin-down} = \left(\frac{H}{\delta_e}\right) f^{-1} =$$

Remember $\quad Ek^{1/2} = \frac{1}{\sqrt{2}} \frac{\delta_e}{H}$

$$T_{spin-down} = \sqrt{2} f^{-1} Ek^{-1/2}$$

$$= \frac{\sqrt{2}}{2\pi} Ek^{-1/2} T_i$$

$T_i = \frac{2\pi}{f}$ an inertial period

For small Ekman numbers, the spin-down time is much longer than an inertial period.

→ It takes a long time for geostrophic flows to spin-down

Notice that it is not the direct action of viscosity that spins-down the geostrophic flow in the interior (since friction is very weak there) it is vortex squashing/stretching caused by the Ekman pumping/suction (which arises directly from frictional effects) that results in the spin-down of the flow. It is by this indirect manner geostrophic flows are decellerated by friction.